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ON *C*-STABILITY OF DIFFERENCE BOUNDARY PROBLEMS OF GENERAL TYPE*

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The stability in space C of linear difference initial boundary-value problems with constant coefficients is investigated. Such problems are usually used to understand the instability cause. The necessary and sufficient C-stability conditions are proved. The principal point of the proof is construction of a simple quasi-Jordan normal form of the resolvent matrix in a neighborhood of $|z| = 1$. The detailed asymptotics of the difference Green functions are constructed by using the saddle point method. The nonlinear instability observed in numerical simulations of fluxon dynamics is studied by using these asymptotics. The problem of stability verification on PC by using computer algebra systems is discussed shortly.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Об устойчивости в *C* разностных краевых задач общего вида

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Исследуется устойчивость в C линейных разностных краевых задач с постоянными коэффициентами. Такие задачи обычно используются для выяснения причин неустойчивости. Доказаны необходимые и достаточные условия устойчивости в C. Принципиальную трудность представляет построение простой квазижордановой нормальной формы резольвентной матрицы в окрестности единичной окружности $|z| = 1$. Методом перевала построены детальные асимптотики разностных функций Грина. С помощью этих асимптотик исследован эффект нелинейной неустойчивости, наблюдавшийся при численном моделировании динамики флюксонов. Коротко обсуждается проблема проверки устойчивости на PC с использованием систем компьютерной алгебры.

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At JINR the finite-difference method is widely used in numerical simulations of physical processes (calculation of electromagnetic fields in accelerators, computing of bound states in the theoretical nuclear physics, studying of wave motion on lattices in condensed matter physics and others). The construction of effective numerical algorithms is connected with the solution of the stability problem. In this work the necessary and sufficient conditions of C-stability for linear boundary difference systems with constant coefficients are presented. Such model problems are used to study the instabilities observed in real calculations. Full theoretical investigation of numerical algorithms for solving

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complicated applied problems, as a rule, is as unreal as solving them without computer. However, when the instability cause is understood, we usually succeed in finding an efficient stable algorithm of solving original applied problem. In this work we discuss studying of two instabilities observed in numerical simulation of fluxon dynamics in one-dimensional Josephson junctions with micro-heterogeneities.

The stability in space C of difference problems

$$\upsilon_\nu^{n+1} = \sum_{l=-rl}^{r2} A_l \upsilon_{\nu+l}^n, \quad n \ge 0, \ \nu \ge 1; \quad \upsilon_\nu^0 = f_\nu, \ |f_\nu| < c, \tag{1}$$

$$\upsilon_m^n = \sum_{j=1}^{s} C_{jm} \upsilon_j^n, \quad m = -r1 + 1, \dots, 0, \tag{2}$$

is discussed. Here υ_j^n are k-dimensional vectors, A_j, C_{jm} are constant matrices. We suppose that $r1 \ge 1$ and $\det(A_{-r1}) \ne 0$, $\det(A_{r2}) \ne 0$. The necessary and sufficient conditions are proved without any additional conditions [1].

Denote by $\mathcal{B}$ the Banach space of semi-infinite vector-sequences $V = \{\upsilon_\nu\}_{\nu \ge -r1}$, satisfying boundary conditions (2). The norm in space $\mathcal{B}$ is simply C-norm:

$$\| V \| = \sup_{\nu \ge 1} (\max_{1 \le i \le k} |\upsilon_\nu(i)|).$$

We rewrite the considered initial boundary-value problem (l), (2) in operator form

$$V^{n+1} = GV^n.$$

G is bounded operator in space $\mathcal{B}$

We use the classic stability definition: *the problem (1), (2) is stable in space C if there exists a positive constant c such that inequality $\| V^n \| \le c \| V^0 \|$ holds for all $n \ge 0$ and all $V^0 \subset \mathcal{B}$*

The necessary stability conditions are well known [2], [3]. The corresponding Cauchy problem must be stable in space C. It's natural here — we consider explicit difference systems. In addition the spectrum of G must lie in the unit disc $|z| \le 1$.

We use the classic spectrum definition: a point z_0 of complex plane is spectrum point of G if there is found nonzero V_0 from $\mathcal{B}$ such that $GV_0 = z_0 V_0$.

If there are spectrum points outside the unit disk, G powers ($\| G^n \|$) grow exponentially when n increasing to infinity.

To get necessary and sufficient stability conditions we present G powers as an integral of the resolvent:

$$G^n = -\frac{1}{2\pi i} \oint_\Gamma (G - zI)^{-1} z^n dz.$$

Here Γ is a closed line rounding all spectrum points of G which are singularities of the resolvent $(G - zI)^{-1}$. We need the explicit form of the resolvent near the unit circle $|z| = 1$. To find explicit form of the resolvent means to find explicit solution of the following eigenvalue problem:

$$\sum_{l=-r1}^{r2} A_l u_{\nu+l} - z u_\nu = f_\nu, \quad \nu \geq 1,$$
$$u_m = \sum_{j=1}^{s} C_{jm} u_j, \quad m = -r1 + 1, \ldots, 0. \tag{3}$$

Here $\{f_\nu\}$ and $\{u_\nu\}$ are given and found sequences, respectively. After the change of variables

$$Y_\nu = [u_{\nu + r2 - 1}, \ldots, u_\nu, \ldots, u_{\nu - r1}]^*$$

($*$ asterix transforms row-vector into column-vector) the system (3) takes the form

$$Y_{\nu+1} = M(z) Y_\nu + F_\nu, \quad \nu \geq 1,$$
$$[0, I] Y_1 = \sum_{l=r1}^{s1} [0, B_l] Y_{l+1}. \tag{4}$$

Here $M(z)$ is a resolvent matrix:

$$M(z) = \begin{bmatrix} -A_{r2}^{-1} A_{r2-1} & \ldots & -A_{r2}^{-1}(A_0 - zI) & \ldots & -A_{r2}^{-1} A_{-r1} \\ I & 0 \ldots & 0 & \ldots 0 & 0 \\ . & . \ldots & . & \ldots . & . \\ 0 & 0 \ldots & 0 & \ldots I & 0 \end{bmatrix},$$

depending linearly on z. The vectors F_ν are defined as follows $F_\nu = [A_{r2}^{-1} f_\nu, 0, \ldots, 0]^*$. And B_l in the boundary condition are constant matrices constituted from the initial constant boundary matrices C_{jm}.

To construct the resolvent we reduce $M(z)$ to the simple quasi-Jordan normal form. $M(z)$ is square matrix of high $(r1 + r2) \cdot k$ order. The problem is solved in general form thanks to the following. Eigenvalues κ of the resolvent matrix and eigenvalues λ of the characteristic matrix

$$D(e^{i\phi}) = \sum_{j=-r1}^{r2} A_j e^{ij\phi}$$

are one-to-one algebraic functions. Knowing the structure of λ we can determine the structure of κ. The basis in which the resolvent matrix takes simple quasi-Jordan normal form is constructed from the corresponding basis for the characteristic matrix.

We present first the quasi-Jordan normal form for the characteristic matrix. When the Cauchy problem is stable the von-Neumann condition is fulfilled: $|\lambda(e^{i\phi})| \le 1$, $0 \le \phi \le 2\pi$. In the case of *C*-stability there can be only a number of isolated points of the unit circle $e^{i\phi_0}$, where some λ take values equal to one in absolute value: $\lambda(e^{i\phi_0}) = e^{i\psi_0}$. We call such points the determining points. If the Cauchy problem is *C*-stable, principal eigenvalues have special expansions near the determining points [4], [5]:

$$\lambda(e^{i\phi}) = \exp\{i\psi_0 + i\gamma(\phi - \phi_0) + (i\alpha - \beta)(\phi - \phi_0)^{2\mu} + O((\phi - \phi_0)^{2\mu + \theta})\}.$$

Here ϕ_0, ψ_0, γ, α are real, β, θ are positive, μ is integer. The principal parts of these expansions

$$\bar{\lambda} = \exp\{i\psi_0 + i\gamma(\phi - \phi_0) + (i\alpha - \beta)(\phi - \phi_0)^{2\mu}\}$$

have only integer powers of $(\phi - \phi_0)$.

With respect to every determining point $e^{i\phi_0}$ eigenvalues of *D* split into the classes

$$\Lambda_0, \Lambda_1, \ldots, \Lambda_\eta, \quad \eta = \eta(\phi_0).$$

The class Λ_0 contains eigenvalues less than one in absolute value in the considered determining point. Main eigenvalues λ_i, λ_j belong to the class Λ_ξ if their principal parts are equal identically $\bar{\lambda}_i \equiv \bar{\lambda}_j$.

It had been proved [6] that, if the Cauchy problem is *C*-stable, the characteristic matrix is reduced (in a neighborhood of the determining points by nonsingular analytic similarity transformation) to the special block-diagonal form $D = \mathrm{diag}\,(A, C_1, \ldots, C_\eta)$. The block *A* has $\lambda \subset \Lambda_0$. Each class Λ_ξ of main eigenvalues has the block C_ξ of simple (Jordan in principal part) structure

$$C_\xi = \bar{\lambda}(I + \Delta N_\xi(\phi - \phi_0)^{2\mu} + (\phi - \phi_0)^{2\mu + 1} R_\xi(\phi)).$$

Here $\bar{\lambda}$ is the principal part of eigenvalues from the considered class Λ_ξ, *I* is the unit matrix, Δ is any constant and $R_\xi(\phi)$ is analytic matrix of general type. All matrices here have only integer powers of $(\phi - \phi_0)$ in their expansions. We call such block-diagonal form «quasi-Jordan normal form». It has the Jordan structure in principal part determining the stability. Now we return to the resolvent matrix.

The Cauchy problem stability implies [2] that for *z* lying outside the unit disk eigenvalues of the resolvent matrix are devided into two nonintersecting sets: there are exactly $r1 \cdot k$ eigenvalues κ less than one in absolute value and exactly $r2 \cdot k$ eigenvalues κ

greater than one in absolute value. When z changing outside the unit disk, eigenvalues κ can't go from one set to another. But when z reaches the unit circle, some κ can take limit values equal to one in absolute value. Point ψ_0 is called the determining point for the resolvent matrix if in this point at least one κ is equal to one in absolute value: $\kappa(e^{i\psi_0}) = e^{i\phi_0}$. In such case ϕ_0 is determining point for the characteristic matrix and for some eigenvalue of the characteristic matrix $\lambda(e^{i\phi_0}) = e^{i\psi_0}$. The expansions of main κ near the determining points are obtained by inversion of the corresponding λ-expansions. It's well known [5] that the difference Green function is located near difference characteristics defined by $\nu + \gamma n = 0$, where $i\gamma$ are coefficients by linear terms in λ-expansions. We call hyperbolic eigenvalues λ with slope characteristics, with $\gamma \neq 0$.

The hyperbolic class Λ_ξ with the principal part

$$\bar{\lambda} = \exp\{i\psi_0 + i\gamma(\phi - \phi_0) + (i\alpha - \beta)(\phi - \phi_0)^{2\mu}\}, \quad \gamma \neq 0,$$

generates the unique hyperbolic κ-class $\mathcal{K}_\xi$ with

$$\bar{\kappa} = \exp\left\{ i\phi_0 + i\frac{\psi - \psi_0}{\gamma} + \frac{\beta - i\alpha}{\gamma^{2\mu+1}}(\psi - \psi_0)^{2\mu} \right\}.$$

As α is real, $\beta > 0$, μ is integer, the hyperbolic λ-classes with negative γ generate κ-classes of eigenvalues less than one in absolute value. And the hyperbolic λ-classes with positive γ generate κ-classes of eigenvalues greater than one in absolute value. We mean here z lying outside the unit disc in a neighborhood of considered determining point $e^{i\psi_0}$.

Each parabolic eigenvalue λ with $\bar{\lambda} = \exp\{i\psi_0 + (i\alpha - \beta)(\phi - \phi_0)^{2\mu}\}$ generates 2μ eigenvalues κ with

$$\bar{\kappa}_l = \exp\left\{ i\phi_0 + i\left(\frac{\psi - \psi_0}{\alpha + i\beta}\right)_l^{\frac{1}{2\mu}} \right\}, \quad l = 1, 2, \ldots, 2\mu.$$

With properly chosen branches the first μ eigenvalues κ_l are less than one in absolute value. And the rest μ ones are greater than one in absolute value. We mean again z lying outside the unit disc in a neighborhood of the considered determining point $e^{i\psi_0}$. So every parabolic Λ_ξ class generates 2μ parabolic κ-classes. The first μ classes $\mathcal{K}^1_{\xi 1}, \ldots, \mathcal{K}^1_{\xi\mu}$ form set $\mathcal{K}^1_\xi$ of eigenvalues less than one in absolute value. The rest μ classes $\mathcal{K}^2_{\xi 1}, \ldots, \mathcal{K}^2_{\xi\mu}$ form set $\mathcal{K}^2_\xi$ of eigenvalues greater than one in absolute value. Now we can present quasi-Jordan normal form of the resolvent matrix [1].

Theorem 1. *If the Cauchy problem is C-stable, and the boundary matrices* A_{-r1}, A_{r2} *are nonsingular, for every determining point* $e^{i\psi_0}$ *there is found analytic nonsingular similarity transformation* $T(\psi)$, $|\psi - \psi_0| < \rho$, *which reduces the resolvent matrix to the following quasi-Jordan normal form:*

$$\hat{M} = T^{-1}MT = \begin{bmatrix} M_{11} & \hat{B} \\ 0 & M_{22} \end{bmatrix} = \begin{bmatrix} M_1 & 0 & 0 & 0 \\ 0 & A & B & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & M_2 \end{bmatrix}, \quad |z - z_0| < \rho, \; z = e^{i\psi}.$$

Block M_{11} *has eigenvalues less than one in absolute value for* $|z| > 1$. *Block* M_{22} *has eigenvalues greater than one in absolute value for* $|z| > 1$. *Block* M_1 *has eigenvalues* $|\kappa| < 1$ *for* $|z - z_0| < \rho$. *Block* M_2 *has eigenvalues* $|\kappa| > 1$ *for* $|z - z_0| < \rho$. *Matrices A, C have block-diagonal structure. The hyperbolic classes* $\mathcal{K}_\xi$ *have blocks of such quasi-Jordan structure*

$$C_\xi = \bar{\kappa}\,(I + \Delta N(\psi - \psi_0)^{2\mu} + (\psi - \psi_0)^{2\mu + 1} R_\xi(\psi)).$$

The hyperbolic classes with $\gamma < 0$ *have the blocks* C_ξ *on the diagonal of A, the hyperbolic classes with* $\gamma > 0$ *have the blocks* C_ξ *on diagonal of C. The sets* $\mathcal{K}_\xi^1$, $\mathcal{K}_\xi^2$ *of the parabolic* κ*-classes have matrix Jordan boxes* C_ξ^1, C_ξ^2 *on the diagonals of A and C respectively:*

$$C_\xi^{1(2)} = \begin{bmatrix} C_{\xi 1}^{1(2)} & I & 0 & \dots & 0 \\ 0 & C_{\xi 2}^{1(2)} & I & \dots & 0 \\ . & . & . & \dots & I \\ 0 & 0 & 0 & \dots & C_{\xi\mu}^{1(2)} \end{bmatrix}.$$

On the diagonals of these matrix Jordan boxes stay quasi-Jordan blocks corresponding to the separate parabolic κ*-classes:*

$$C_{\xi i}^{1(2)} = \bar{\kappa}_{i(i+\mu)}(I + \Delta N(\psi - \psi_0)_{i(i+\mu)}^{1/2\mu} + (\psi - \psi_0)_{i(i+\mu)}^{(1/2\mu)+\theta} R_{\xi i}^{1(2)}(\psi)), \quad i = 1, \dots, \mu.$$

Their principal parts have Jordan, structure again. But this time there are fractional powers of $(\psi - \psi_0)$. *On the whole big matrix Jordan box* C_ξ *corresponds to the set of parabolic eigenvalues* κ *generated by the parabolic class* Λ_ξ *(with* $\gamma = 0$*):*

$$C_\xi = \left[\begin{array}{ccccc|ccccc} C^1_{\xi 1} & I & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & C^1_{\xi 2} & I & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ . & . & . & \dots & I & . & . & . & \dots & 0 \\ 0 & 0 & 0 & \dots & C^1_{\xi\mu} & I & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & 0 & \dots & 0 & C^2_{\xi 1} & I & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & C^2_{\xi 2} & I & \dots & 0 \\ . & . & . & \dots & . & . & . & . & \dots & I \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & C^2_{\xi\mu} \end{array}\right].$$

The block C^1_ξ belongs to A, the. blocks C^2_ξ belongs to C. The rest nonzero north-east block belongs to B. The block B contains the unit matrices coupling the boundary blocks $C^1_{\xi\mu}$ and $C^2_{\xi 1}$. On other places of B zeros stay.

Remark that there is complete analogy of constructed quasi-Jordan analytic normal form of the resolvent matrices with the canonical Jordan form of the constant matrices. The role of the eigenvalues is played by the quasi-Jordan blocks C_ξ, $C^1_{\xi i}$, $C^2_{\xi i}$ corresponding to κ-classes. The role of Jordan boxes is played by the matrix Jordan boxes C_ξ corresponding to the sets of κ generated by the parabolic λ-classes Λ_ξ (with $\gamma = 0$).

When the system has only slope characteristics, the block B is zero matrix. So the resolvent matrix is factored completely. All matrices have only integer powers $(\psi - \psi_0)$ in their expansions near the determining points ψ_0. When the system has vertical characteristics, the complete factorization is impossible. The presence of fractional powers is inevitable.

After the quasi-Jordan normal form of the resolvent matrix is constructed, we can present the explicit form of the resolvent for z lying near the unit circle. The explicit solution of the system (4) is

$$w^1_\nu = M^{\nu-1}_{11} w^1_1 + \sum_{\xi=1}^{\nu-1} (M^{\nu-1-\xi}_{11} \hat{B} M^{\xi-1}_{22}) w^2_1 + \sum_{\xi=1}^{\nu-1} M^{\nu-1-\xi}_{11} [(T^{-1}F_\xi)^1 +$$

$$+ \hat{B} \sum_{\eta=1}^{\xi-1} M^{\xi-1-\eta}_{22} (T^{-1}F_\eta)^2],$$

$$w^2_\nu = M^{\nu-1}_{22} w^2_1 + \sum_{\xi=1}^{\nu-1} M^{\nu-1-\xi}_{22} (T^{-1}F_\xi)^2, \quad \nu \geq 2,$$

$$w_1^2 = -\sum_{\xi=1}^{\infty} M_{22}^{-\xi}(T^{-1}F_\xi)^2,$$

$$K_1 w_1^1 = K_2 w_1^2 + \sum_{\xi=1}^{s1} [B_\xi^1 (T^{-1}F_\xi)^1 + B_\xi^2 (T^{-1}F_\xi)^2].$$

Here w_ν^1 and w_ν^2 are vectors composed of the top $r1\cdot k$ and bottom $r2\cdot k$ components of the vector $W_\nu = T^{-1}Y_\nu$, and K_1 and K_2 are analytic matrices

$$K_1 = T_{21} - \sum_{\xi=r1}^{s1} B_\xi T_{21} M_{11}^{\xi},$$

$$K_2 = -T_{22} + \sum_{\xi=r1}^{s1} B_\xi (T_{22} M_{22}^{\xi} + T_{21} \sum_{\eta=1}^{\xi-1} M_{11}^{\xi-1-\eta} \hat{B} M_{22}^{\eta-1}),$$

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix},$$

T_{11}, T_{22} are square matrices of orders $r1\cdot k$, $r2\cdot k$, respectively. It's clear now that the resolvent singularities are singularities of $K_1(z)$. It had been proved [2] that z_0 belongs to the G-spectrum iff $\det K_1(z_0) = 0$, $|z_0| \geq 1$.

To simplify the stability theorem formulation we consider in addition two singularity matrices S_1, S_2.

To get S_1 we replace in the block M_{11} the block M_1 and all hyperbolic blocks C_ξ by the unit matrices. In the parabolic matrix Jordan boxes C_ξ^1 we replace the diagonal blocks $C_{\xi i}^1$ by the singular diagonal matrices $(\psi - \psi_0)^{-(i-1)/2\mu} I$ and remove all unit matrices on the next diagonal — all Jordan connections are replaced by zero matrices.

To get S_2 we replace in the block M_{22} the block M_2 by the unit matrix. All hyperbolic blocks C_ξ are replaced by the singular matrices $(\psi - \psi_0)^{-1} I$. In the parabolic matrix Jordan boxes C_ξ^2 we replace the diagonal blocks $C_{\xi i}^2$ by the singular diagonal matrices $(\psi - \psi_0)^{-(\mu - i + 1)/2\mu} I$ and again remove all Jordan connections. Now we formulate the stability theorem [1].

Theorem 2. *The initial baundary-value problem (1), (2) is C-stable iff conditions 1.2,3 are satisfied. The first two are simply the necessary conditions.*

1. The corresponding Cauchy problem must be C-stable [5].

2. The spectrum of G must lie in the unit disc $|z| \leq 1$.

The third condition refers to the spectrum points lying on the unit circle.

3. At the spectrum points z_0, $|z_0| = 1$, *the matrices*

$$S_1 K_1^{-1}(z), \quad S_1 K_1^{-1}(z) K_2(z) S_2$$

must have no singularities of order higher than the first. In the case of hyperbolic systems (when the elements of the quasi-Jordan normal form of the resolvent matrix have only integer powers in their expansions and $S_1 = I$*) matrices* $K_1^{-1}(z)$, $K_1^{-1}(z)K_2(z)S_2$ *can have singularities of the first order only. When there are vertical characteristics (there are parabolic eigenvalues), fractional powers appear. So we say in general case «no singularities of order higher than the first».*

If the conditions 1,2 are satisfied but the condition 3 is violated, then $\| G^n \| \asymp n^s$ *when* $n \to \infty$*. Here s is a maximum of diviations of admissible singularity orders over all spectrum points lying on the unit circle.*

The starting point for our research was remarkable work [2] of H.-O.Kreiss, where the sufficient conditions of L_2-stability for considered initial boundary-value problems (1), (2) had been proved.

In my doctor dissertation [7], necessary and sufficient conditions of L_2-stability for (1), (2) have been proved. Principally new was the analysis of the spectrum points lying on the unit circle. In the case of instability, the precise in order power estimates in L_2

$$\| G^n \| \asymp n^s, n \to \infty,$$

have been proved. Only hyperbolic systems were analysed. Here we discuss C-stability for the systems of general type.

Why do we need stability in space C? It's well known that in numerical simulations the Gibbs phenomenon is observed near discontinuities. It had been proved theoretically ([8], [[9], [10] and others) that in the case of G-stability weak exponentially decreasing oscillations arise. If there is L_2-stability only, the oscillations perturbing the solution essentially develop near discontinuities.

Besides, when the high accuracy schemes are used, we need additional boundary conditions. And we must choose such additional boundary conditions that distortion of the solution were as small as possible. It had been proved theoretically ([11], [12] and others) that when the problem with additional boundary conditions is stable in space C, the solution «feels» these additional boundary conditions in the boundary layer of $O(\ln N)$ length only. Here N is the number of points in the unit interval.

The proof of the **Theorem I** and **Theorem 2** contains algorithm for stability verification, which can be realized on PC by using computer algebra systems. But hereby a lot of new difficulties arise. In the process of proving, detailed asymptotics of the difference Green functions were obtained [7], [13] by using the saddle point method. We mean asymptotics of the integrals

$$\oint_{\Gamma} (z - z_0)^{-\xi} M_{11}^{\nu}(z) z^n dz, \quad \nu \geq 1, \quad n \to \infty.$$

Here ξ is a positive constant, M_{11} is the block of the resolvent matrix. These asymptotics can be used in studying problems with variable coefficients [14]. They can be useful as well in studying nonlinear effects of instability. In the rest part we discuss two effects of instability observed in numerical simulation of fluxon dynamics [15].

The fluxon motion in long Josephson junctions with micro-inhomogeneities is described by the sin-Gordon equation with singularities in coefficient by sin (ϕ):

$$\phi_{tt} = \phi_{xx} - \left(1 - \sum_{i=1}^{s} \mu_i \delta(x - x_i)\right) \sin(\phi), \quad -l \le x \le l,$$

$$\phi_x(-l) = \phi_x(l) = 0.$$

«The well formed fluxons» are used as initial data. For the sake of simplicity only, we consider the unique micro-inhomogeneity in x_0 in what follows. When $\phi(x_0) \neq k\pi$, ϕ_x has discontinuity in x_0:

$$\phi_x(x_0 + 0) - \phi_x(x_0 - 0) = -\mu \sin \phi(x_0).$$

An energy relation

$$\frac{d\mathcal{E}}{dt} = \frac{d}{dt} \int_{-l}^{l} \left[\frac{\phi_t^2 + \phi_x^2}{2} + (1 - \cos \phi) \right] dx - \mu(1 - \cos \phi(x_0, t)) = 0$$

holds. It says that energy $\mathcal{E}$ must stay constant.

The problem is solved numerically [15]. Beforehand we replace the sin-Gordon equation by equivalent integro-differential system with respect to new variables $u = \phi_x$, $\upsilon = \phi_t$. The differential part of the system is approximated by the Wendroff-Lax scheme or the Rusanov scheme, delta function is approximated by $h^{-1}\delta_{\nu\nu_0}$. Here $\delta_{\nu\nu_0}$ is the Kronecker symbol. The area of rectangle of the height h^{-1} with interval $[x_0 - h/2, x_0 + h/2]$ as foundation is equal to 1. The discontinuity is well seen in numerical solution but it's eroded. The distortion of $\phi(x_0)$ value leads to violation of energy balance. In some cases «the numerical energy» $\tilde{\mathcal{E}}$ decreases and the fluxon is destroyed at the end.

To understand the nature of observed discontinuity erosion, we consider the following model problem:

$$u_t = u_x - \mu\delta(x) \sin(u), \quad -\infty < x < \infty,$$

$$u(x, 0) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x \ge 0. \end{cases}$$

This is nonlinear equation with δ-function in the coefficient by sin (u) again. The solution of this problem is the sum of stationary step

$$u_1(x,\mu)=\begin{cases}0 & \text{for } x<0,\\ \mu\sin 1 & \text{for } x\ge 0,\end{cases}$$

of height $\mu \sin 1$ and a step of the height $(1-\mu\sin 1)$

$$u_2(t,x,\mu)=\begin{cases}0 & \text{for } x+t<0,\\ 1-\mu\sin 1 & \text{for } x+t\ge 0.\end{cases}$$

with discontinuity moving along the characteristic of the primitive hyperbolic equation $u_t=u_x$. By using the detailed asymptotics of the difference Green function and the difference step function for this primitive hyperbolic equations we got [16] asymptotics of exact solution of discrete approximations for the considered nonlinear model equation with singularity in coefficients by $\sin(u)$. The found asymptotics say that the solutions of the discrete problems approximate the solution of the considered model problem with wrong parameter μ values:

$$u_1(x,\tilde{\mu})+u_2(t,x,\tilde{\mu}),\quad \tilde{\mu}\ne\mu.$$

In order the solutions of the discrete problems approximate the right solution, we must use in discrete problems fictitious μ^* instead of μ. For the Wendroff-Lax approximation

$$\mu^*=\frac{\mu\sin 1}{\sin(1-\mu\sin 1)}$$

and for the Rusanov approximation

$$\mu^*=\frac{\mu\sin 1}{\sin(1-\mu(1+c_0)\sin 1)}.$$

Here c_0 depends on the parameters of the scheme. Besides standard parameter $\alpha=\tau/h$ the Rusanov scheme of the third-order accuracy has additional parameter ω which does not influence the accuracy but gives possibility to rule the stability. The Rusanov scheme is stable in space C if (α,ω) satisfy inequalities $0<\alpha<1$, $4\alpha^2-\alpha^4<\omega<3$. For $\alpha=1/2$, $\omega=2$ (point from the C-stability region) we found that

$$c_0=\frac{-24d}{(1-d)(3d^2+27)},$$

where d is unique real solution of $z^3+27z-4=0$. So $c_0=-0.1540....$ We verify the μ^* values found theoretically by the right calculations. In both the cases (the Wendroff-Lax and the Rusanov approximations) we had perfect result.

In numerical simulations of fluxon dynamics we observed another instability developing in the boundary region. The Rusanov scheme has 5 points in the low layer. So we need additional boundary conditions. A number of second-order approximations were used for this. And we observed steadily strong oscillations near boundary points. At last, when the Wendroff-Lax approximation was used as additional boundary conditions, the strong oscillations died down. The numerical experiments showed that the same strong oscillations arise in the case of the wave equations. And this is natural because the singularity of the considered sin-Gordon equation is at the finite distance from the boundary points. The oscillations develope before the singularity influences the solution in the boundary region. So we can use the simple wave equation for studying observed boundary instability.

We simply computed [17] by using the REDUCE system the spectra of two initial boundary-value problems. In the first problem, one of the second-order approximations leading to the strong oscillations was chosen as additional boundary condition. In the second problem, the Wendroff-Lax approximation was used. The spectrum point $z_0 = -1.067...$ was found for the fist problem. This spectrum point lying very close to the unit circle leads to the strong-oscillations. For the second problem the unique spectrum point $z_0 = 0$ was found. In both the cases the calculations were performed with $\alpha = 1/2$, $\omega = 2$.

The spectrum of the considered problem is described by a polynomial system

$$\begin{cases} \det R(z, \kappa_1, \kappa_2, \kappa_3, \kappa_4) = 0, \\ P_8(z, \kappa_i) = 0, \quad i = 1, 2, 3, 4, \quad |\kappa_i(z)| < 1, \quad |z| > 1, \end{cases}$$

κ_i are the roots of the characteristic polynomial of the 8th order with coefficients depending on z. The elements of the i-th column of R matrix depend on z and κ_i only. We tried to solve the considered polynomial system by the right method of REDUCE system without success. After a number of elementary manipulations with the columns of R we get new determinant equation $\det \hat{R}(x, y, z) = 0$. Elements of $\hat{R}$ matrix are polynomials of $x = \kappa_3^{-1} + \kappa_4^{-1}$, $y = \kappa_3\kappa_4$, z. The symmetry of the roots of the characteristic polynomial gives possibility to exclude κ_1, κ_2. By using the Vieta relations in addition we found that x. z can be presented as simple functions of y

$$x = \frac{1 - 31y}{1 - 4y^2}, \quad z = \frac{4y^2 + x(1 - x)y + 21y + 1}{48y}.$$

So the problem of spectrum calculation is reduced here to the solution of the unique polynomial of the 18th order with big coefficients

$$Q(y) = \sum_{i=0}^{18} a_j y^j, \quad a_{18} = -8978432,...$$

Such polynomials can't be computed and solved by hands. We solved the problem on PC by using REDUCE system [17]. It was the first success. Then an algorithm for stability verification on PC by using computer algebra systems was developed. We studied by this algorithm a number of model problems of practical interest.

In particular, nontrivial difference problem with initial, additional boundary and overlap conditions were studied [18], [19], [20]. Such problem appears, for instance, when different overlap grids are used. The Rusanov scheme and the Gary scheme used in airflow simulations were studied. In the case of the Gary scheme the spectrum is described by a system of 7 polynomial equations. The problem is reduced to the solution of the unique polynomial equation of 924 order with huge coefficients [20]. This polynomial was computed and factored on PC 486-66/16mb by using MAPLE system in 20 minutes approximately. We get 9 different polynomials of order not more than 100. These polynomials were solved by using REDUCE system. In the case of the grids without displacement (with proper parameters) the unique in $|z| \geq 1$ spectrum point $z_0 = 1$ was found. This leads to the power instability in space L_2: $\| G^n \| \asymp \sqrt{n}$. Simple displasement provides stability. This analytical results are in agreement with the results of numerical experiments [21]. We studied an example of interesting instability phenomenon: the instability is observed in computations only for even number of full steps on the overlap interval for the outflow problem.

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